ON THE EXISTENCE OF BALANCED AND SKT METRICS ON NILMANIFOLDS

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Abstract. On a complex manifold an Hermitian metric which is simultaneously SKT and balanced has to be necessarily Kähler. It has been conjectured that if a compact complex manifold (M, J) has an SKT metric and a balanced metric both compatible with J, then (M, J) is necessarily Kähler. We show that the conjecture is true for nilmanifolds.

1. Introduction

A Riemannian metric g on a complex manifold (M, J) is *compatible* with J (or J-*Hermitian*) if $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. In the present paper we focus on the existence of *special* Hermitian metrics on complex manifolds. More precisely, we study the existence of *strong Kähler with torsion* (shortly SKT) and balanced metrics compatible with the same complex structure. We recall that a J-Hermitian metric is called *SKT* (or *pluriclosed*) if its fundamental form ω satisfies

$$\partial\bar{\partial}\omega = 0\,,$$

while g is called *balanced* if ω is co-closed, i.e.

$$d^*\omega = 0,$$

where d^* denotes the formal adjoint operator of d with respect to the metric g.

SKT metrics were introduced by Bismut in [6] and further studied in many papers (see e.g. [13, 14, 11, 20, 18, 19, 22] and the references therein), while balanced metrics were introduced and firstly studied by Michelsohn in [17], where their existence is characterized in terms of currents. In a subsequent paper Alessandrini and Bassanelli proved that modifications of compact balanced manifolds are always balanced (see [1, 2]) showing a powerful tool for finding examples of balanced manifolds.

It is well-known that if an Hermitian metric g is simultaneously SKT and balanced, then it is necessarily Kähler (see e.g. [4]). This result has been generalized in [15] showing that a compact SKT conformally balanced manifold has to be Kähler. The next conjecture was stated in [13] and it is about the existence of

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an SKT metric and a balanced metric both compatible with the same complex structure:

Conjecture. *Every compact complex manifold admitting both an SKT metric and a balanced metric is necessarily Kähler.*

The conjecture has been implicitly already proved in literature in some special cases. For instance, Verbitsky has showed in [22] that the twistor space of a compact, anti-self-dual Riemannian manifold has no SKT metrics unless it is Kählerian and Chiose has obtained in [7] a similar result for non-Kähler complex manifolds belonging to the Fujiki class. Furthermore, Li, Fu and Yau have proved in [20] that some new examples of SKT manifolds do not admit any balanced metric. A natural source of non-Kähler manifolds admitting balanced metrics and SKT metrics are given by *nilmanifolds*, i.e. by compact manifolds obtained as quotients of a simply connected nilpotent Lie group G by a co-compact lattice Γ. It is well known that a nilmanifold cannot admit Kähler structures unless it a torus (see for instance [5, 12]). The aim of the paper is to show that the conjecture is true when (M, J) is a *complex nilmanifold*. By *complex nilmanifold* we refer to a nilmanifold equipped with an *invariant* complex structure J, i.e. endowed with a complex structure induced by a left-invariant complex structure on G.

Our result is the following

Theorem 1.1. *Let $M = G\backslash\Gamma$ be a nilmanifold equipped with an invariant complex structure J. Assume that (M, J) admits a balanced metric g and an SKT metric g' both compatible with J. Then (M, J) is a complex torus.*

The theorem is trivial in dimension 6 in view of the main result in [10] and it was already proved in [13] when the nilmanifold has dimension 8 by using a classification result proven in [8].

2. Proof of Theorem 1.1

In order to prove Theorem 1.1 we need the following lemmas

Lemma 2.1. *Let $(M = G/\Gamma, J)$ be a complex nilmanifold.*

- *If (M, J) has a balanced metric, then it has also an* invariant *balanced metric* [9].
- *If (M, J) has an SKT metric, then it has also an* invariant *SKT metric* [21].

Lemma 2.2 ([8])**.** *Let $(M = G/\Gamma, J)$ be a complex nilmanifold of real dimension $2n$. If (M, J) has an SKT metric, then G is (at most) 2-step nilpotent, and there exists a complex $(1,0)$-coframe $\{\alpha^1, \dots, \alpha^n\}$ on $\mathfrak{g}$ satisfying the following structure equations*

$$\begin{cases} d\alpha^j = 0, \quad j = 1, \dots, k, \\ d\alpha^j = \sum_{r,s=1}^{k} \left(\frac{1}{2} c^j_{rs} \alpha^r \wedge \alpha^s + c^j_{r\bar{s}} \alpha^r \wedge \bar{\alpha}^s\right), \ j = k+1, \dots, n\,, \end{cases}$$

for some $k \in \{1, \dots, n-1\}$ and with $c^j_{rs}, c^j_{r\bar{s}} \in \mathbb{C}$.

Now we can prove Theorem 1.1

Proof of Theorem 1.1. Suppose that $(M = G/\Gamma, J)$ is not a complex torus, i.e. that G is not abelian and denote by $\mathfrak{g}$ the Lie algebra of G. Assume that (M, J) admits a balanced metric g and also an SKT metric g' both compatible with J. Then in view of Lemma 2.1, we may assume both g and g' invariant and regarding them as scalar products on $\mathfrak{g}$. This allows us to work at the level of the Lie algebra $\mathfrak{g}$. As a consequence of Lemma 2.2, the existence of the SKT metric implies that Lie algebra $\mathfrak{g}$ is 2-step nilpotent and that $(\mathfrak{g}, J)$ has a $(1,0)$-coframe $\{\alpha^1, \ldots, \alpha^n\}$ satisfying the following structure equations

$$\begin{cases} d\alpha^j = 0, \quad j = 1, \ldots, k, \\ d\alpha^j = \sum_{r,s=1}^{k} \left(\frac{1}{2} c^j_{rs}\alpha^{rs} + c^j_{r\bar{s}}\alpha^{r\bar{s}}\right). \end{cases} \tag{1}$$

for some $k \in \{1, \ldots, n-1\}$ and with $c^j_{rs}, c^j_{r\bar{s}} \in \mathbb{C}$. Here we use the notation $\bar{\alpha}^i = \alpha^{\bar{i}}$ and $\alpha^{r_1\cdots r_p\bar{s}_1\cdots\bar{s}_q} = \alpha^{s_1} \wedge \ldots \alpha^{s_p} \wedge \alpha^{\bar{s}_1} \wedge \ldots \alpha^{\bar{s}_q}$. We may assume without restrictions that the coframe $\{\alpha^i\}$ is unitary with respect to the balanced metric g. Indeed, since in the Gram-Schmidt process the spaces spanned by the first r elements of the original basis are preserved, we can modify the coframe $\{\alpha^r\}$ making it unitary with respect to g and satisfying the same structure equations as in (1) with different structure constants. In this way g writes as

$$g = \sum_{r=1}^{n} \alpha^r \otimes \alpha^{\bar{r}}$$

and the balanced condition can be written in terms of $c^l_{r\bar{s}}$'s as

$$\sum_{r=1}^{k} c^l_{r\bar{r}} = 0, \tag{2}$$

for every $l > k$ (see also [3, Lemma 2.1]).

Next we focus on the SKT metric g'. Since $\mathfrak{g}$ is 2-step nilpotent, we have $\partial\bar{\partial}\alpha^r = 0$, $r = 1, \ldots, n$. If

$$\omega' = \sum_{i,j=1}^{n} a_{i\bar{j}}\alpha^i \wedge \alpha^{\bar{j}}$$

is the fundamental form of g', then the SKT condition $\partial\bar{\partial}\omega' = 0$ writes as

$$\sum_{i,j=k+1}^{n} a_{i\bar{j}}(\overline{\partial}\alpha^i \wedge \partial\alpha^{\bar{j}} - \partial\alpha^i \wedge \overline{\partial}\alpha^{\bar{j}}) = 0\,. \tag{3}$$

Equation (3) can be written in terms of the structure constants as

$$\sum_{i,j=k+1}^{n} \sum_{r,s,u,v=1}^{k} a_{i\bar{j}} \left(c^i_{r\bar{v}}\bar{c}^j_{u\bar{s}} + \frac{1}{4}c^i_{rs}\bar{c}^j_{uv}\right)\alpha^{rs\bar{u}\bar{v}} = 0.$$

By considering the component along $\alpha^{rs\bar s\bar r}$ in the above expression we get

$$\sum_{i,j=k+1}^{n} a_{i\bar j}\left(c^i_{r\bar r}\bar c^j_{s\bar s}+\frac14 c^i_{rs}\bar c^j_{sr}-c^i_{s\bar r}\bar c^j_{s\bar r}-\frac14 c^i_{sr}\bar c^j_{sr}+c^i_{s\bar s}\bar c^j_{r\bar r}+\frac14 c^i_{sr}\bar c^j_{rs}-c^i_{r\bar s}\bar c^j_{r\bar s}-\frac14 c^i_{rs}\bar c^j_{rs}\right)=0\,,$$

i.e. the condition

$$\sum_{i,j=k+1}^{n} a_{i\bar j}\left(c^i_{r\bar r}\bar c^j_{s\bar s}-c^i_{s\bar r}\bar c^j_{s\bar r}+c^i_{s\bar s}\bar c^j_{r\bar r}-c^i_{r\bar s}\bar c^j_{r\bar s}+c^i_{rs}\bar c^j_{sr}\right)=0\,. \tag{4}$$

Now by taking the sum of (4) for $r,s=1,\dots k$ and keeping in mind the balanced assumption (2) we get

$$\sum_{i,j=k+1}^{n}\sum_{r,s=1}^{k} a_{i\bar j}\left(2c^i_{s\bar r}\bar c^j_{s\bar r}+c^i_{rs}\bar c^j_{rs}\right)=0\,. \tag{5}$$

Let us consider now the $(1,0)$-vectors X_{sr} and $X_{r\bar s}$ on $\mathfrak{g}$ defined by

$$X_{rs}=\sum_{i=k+1}^{n} c^i_{rs}X_i\,,\quad X_{r\bar s}=\sum_{i=k+1}^{n}\sqrt{2}\,c^i_{r\bar s}X_i$$

where $\{X_1,\dots,X_n\}$ is the dual frame to $\{\alpha^1,\dots,\alpha^n\}$. We have

$$\sum_{r,s=1}^{k}\left(\omega'(X_{rs},\bar X_{rs})+\omega'(X_{r\bar s},\bar X_{r\bar s})\right)=\sum_{i,j=k+1}^{n}\sum_{r,s=1}^{k} a_{i\bar j}\left(2c^i_{s\bar r}\bar c^j_{s\bar r}+c^i_{rs}\bar c^j_{rs}\right)$$

and so equation (5) implies $X_{rs}=X_{r\bar s}=0$ for every $r,s=1,\dots,k$. So every α^r is closed and $\mathfrak{g}$ is abelian, contradicting the first assumption in the proof. □

Remark 2.3. Theorem 1.1 is trivial in dimension 6. Indeed, in view of [10] and Lemma 2.1, if a 6-dimensional complex nilmanifold (M,J) has an SKT metric, then every invariant J-Hermitian metric on (M,J) is SKT and so every invariant balanced metric compatible with J is automatically Kähler. More generally, the same argument works when M has arbitrary real dimension $2n$, but $k=n-1$.

Remark 2.4. Another case where Theorem 1.1 is trivial is when $k=1$. In this case the Lie algebra $\mathfrak{g}$ is necessary isomorphic to $\mathfrak{h}_3^{\mathbb{R}}\oplus\mathbb{R}^{2n-3}$, where $\mathfrak{h}_3^{\mathbb{R}}$ is the real 3-dimensional Heisenberg algebra, and the result follows.

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